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COMMUTATIVITY DEGREE AND GRAPHS RELATED TO CONJUGACY CLASSES OF SOME NON-ABELIAN GROUPS

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Abstract: Let G be a finite group. The commutativity degree of G is the probability that two randomly chosen elements of the group commute. This paper explores the commutativity degrees and the properties of graphs relating to conjugacy classes associated with various group and group products, focusing on dihedral, generalized quaternion, and symmetric groups. We find that the conjugacy class graph of $Q_{4n} \times Q_{4m}$ and $S_n \times S_m$ are connected and non-planar. Furthermore, we examine the generalized conjugacy class graphs of generalized dihedral and generalized quaternion group, providing insights into their graph structures and connectivity properties. Our findings highlight the intricate relationships between group elements, their conjugacy classes, and the resulting graph-theoretic representations.

Keywords and Phrases: Conjugacy class graph, generalized conjugacy class graph, non-abelian group, dihedral group, generalized quaternion group, symmetric group.

2020 Mathematics Subject Classification: 05C10, 05C25.

1. Introduction

The study of finite groups and their properties is a fundamental topic in algebra, with significant applications in various fields. One intriguing aspect of group theory is the commutativity degree, which measures the likelihood that two randomly chosen elements of a group commute. This probability helps in classification of non-abelian groups and provides valuable insights into the structure and behaviour of the group. The concept of commutativity degree in finite groups was introduced by Erdős and Turán [4] in their 1968 paper titled "On Some Problems of a Statistical Group Theory II". They stated that the probability that an arbitrary element $a \in G$ commutes with another arbitrary $b \in G$ is, $P(G) = |\{(a,b) \in G \times G : ab = ba\}|/|G|^2$. This laid the foundation for further studies in the probabilistic aspect of group theory, particularly in understanding the commutative properties of group elements. Notable works include studies by Gustafson and MacHale [7, 12] where they independently showed that this probability can be computed using conjugacy classes and that the probability is less than or equal to 5/8 for finite non-abelian groups. Another contribution is by Omer et. al. [13] where they study and obtain the probability that a group element fixes a set of size two for some finite non-abelian metacyclic 2-groups. They determined that the probability that an element of metacyclic 2-group of nilpotency of class two and at least three is less than or equal to one.

A particularly productive area of research involves the study of conjugacy class graphs which was introduced by Bertram et. al. [1] in 1990, where the graph consists of vertices which are the conjugacy classes of a group and two vertices a, b are adjacent if $gcd(|a|, |b| > 1)$. This graph provides insights into the structure of groups by representing the interactions between their conjugacy classes, revealing patterns and properties that might not be immediately apparent through traditional methods. Another interesting graph related to groups is the generalized conjugacy class graph, in which the vertices are the non-central orbits and two vertices are adjacent if the gcd of the orders of the vertices is greater than 1. This graph was first introduced by Omer et. al. [14] in 2015 and has since been studied extensively. Many researchers are interested in determining the conjugacy class graph and generalized conjugacy class graph of specific groups. For instance, Sarmin et. al. [16] in 2016 computed the conjugacy classes of metacyclic 2-groups of order at most 32 and subsequently studied the properties of the resulting conjugacy class graph. Zulkarnain et. al. [20] in 2020 determined the conjugacy class graphs of some non-abelian 3-groups and found them to be complete graphs with chromatic number, dominating number, and diameter equal to eight, one, and one, respectively. Zamri et. al. [19] in 2017 determined that the generalized conjugacy class graph associated with metacyclic 3-groups and metacyclic 5-groups are complete and null graphs. Additional works in this field can be found in [2, 9, 15, 18].

2. Preliminary Definitions

In this section we state some basic definitions which have been referred to for establishing our main results.

Definition 2.1. Conjugate elements and Conjugacy class [5]
Two elements $a, b \in G$ of a group G are said to be conjugate if there exists an element $c \in G$ such that $b = c^{-1}ac$. The set of all elements conjugate to a is called the conjugacy class of a and is denoted by $Cl(a)$. The conjugacy relation is an equivalence relation that partitions G into disjoint equivalence classes, called the conjugacy classes.

Definition 2.2. Connected graph [3]
A graph G is said to be connected if there exists a path between any two distinct vertices in the graph; otherwise G is said to be disconnected.

Definition 2.3. Planar Graph [3, 8]
A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn in such a way that no edges cross each other. A graph is said to be a planar graph if and only if it contains no subgraph isomorphic to $K_{3,3}$ or K_5.

Definition 2.4. Diameter of a graph [3]
The diameter of a connected graph is the maximum distance between any two vertices in the graph.

Definition 2.5. Chromatic number of a graph [3]
The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of colors required to color the vertices of G so that no two adjacent vertices share the same color.

Definition 2.6. Clique and Clique number of a graph [3]
A clique of a graph G is a complete induced subgraph of G and the number of vertices in the largest clique of G is called the clique number of G.

Definition 2.7. Independence number of a graph [3]
The independence number α of a graph is the maximum number of vertices from the vertex set of the graph such that no two vertices are adjacent.

Definition 2.8. Direct product of two groups [5]
For two groups G_1 and G_2, the direct product $G_1 \times G_2$ is defined as,

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$

Two elements (g_1, h_1) and (g_2, h_2) are conjugate in $G \times H$ if and only if g_1 and g_2 are conjugate in G and h_1 and h_2 are conjugate in H. Thus, each conjugacy class *in $G \times H$ is the cartesian product of a conjugacy class in G and a conjugacy class in H.*

Definition 2.9. Conjugacy class graph [1]
The conjugacy class graph of G is a graph whose vertices are the non-central conjugacy classes of G and two vertices a, b are adjacent if $gcd(|a|, |b|)$ is greater than one.

In this paper, the conjugacy class graphs are represented as follows:
The conjugacy class graphs are such that each vertex represents a set of vertices with same order and adjacent to each other. The adjacency between any two pair of vertices indicates that every point in a vertex is adjacent to each point in the other vertex and so on. Conjugacy classes of same order, say n, are represented by a vertex indexed $[n]$.

Definition 2.10. Orbit and the set Ω [6, 18]
Let G be a finite group. The set Ω is the set of all pairs of commuting elements of the form (a, b), where $a, b \in G$ and the lowest common multiple of the orders of the elements is two. We define Ω as follows:

$$\Omega = \{(a, b) \in G \times G | ab = ba, a \neq b, lcm(|a|, |b|) = 2\}$$

If G acts on a set Ω by conjugation and $x \in \Omega$, then the orbit of x, denoted by $O(x)$, is the subset defined by:

$$O(x) = \{gxg^{-1} | g \in G, x \in \Omega\}$$

The number of orbits is denoted by $K(\Omega)$

Definition 2.11. Generalized conjugacy class graph [14]
Let G be a finite non-abelian group and G acts on Ω by conjugation. The vertices of generalized conjugacy class graph of G, denoted by $\Gamma_G^{\Omega_c}$ are the non-central orbits and two vertices are adjacent if the greatest common divisor of the orders of the vertices is greater than one.

Definition 2.12. Commutativity degree [13]
Let G be a finite group and $K(G)$ is the number of conjugacy classes of the group. Then the commutativity degree of G, denoted by $P(G)$, is defined by:

$$P(G) = \frac{K(G)}{|G|}$$

This concept of commutativity degree was extended to find the probability that an element of a group fixes a set and is defined as follows:

Let G be a group and S be the set of all commuting elements of size two of the form (a, b) in G and G acts on S by conjugation. Then the probability that an element of the group fixes the set S is given by:

$$P_G(S) = \frac{K}{|S|}$$

where K is the number of conjugacy classes (or orbits) of S in G.

3. Conjugacy Class Graph of $Q_{4n} \times Q_{4m}$ and $S_n \times S_m$

This paper is in continuation with [10, 11] where we have discussed the conjugacy class graphs of D_n, Q_{4n}, S_n and $D_n \times D_m$ and their related graph properties. In this section, we explore the graph structures associated with conjugacy classes of $Q_{4n} \times Q_{4m}$ and $S_n \times S_m$.

Before presenting our main results, we now introduce a useful lemma for deriving the conjugacy classes of the generalized quaternion group.

Lemma 1. [17] *The number of conjugacy classes of Q_{4n} is $n+3$ and the conjugacy classes are $\{e\}$, $\{a^n\}$, $\{a, a^{2n-1}\}$, $\{a^2, a^{2n-2}\}$, ... , $\{a^{n-1}, a^{n+1}\}$, $\{ab, a^3b, \ldots, a^{2n-1}b\}$, $\{b, a^2b, a^4b, \ldots, a^{2n-2}b\}$.*

Proof. The generalized quaternion group is defined as:

$$Q_{4n} =< a, b : a^{2n} = e = b^4, a^n = b^2 = (ab)^2, b^{-1}ab = a^{-1} >$$

Here, $a^{2n} = e \implies a^{2n} \in Z(Q_{4n})$. Therefore, $[a^{2n}] = \{a^{2n}\}$; $[a^{2n}]$ is the conjugacy class of a^{2n}.

Let us consider an element $a^n \in Q_{4n}$.

Let $a^j (1 \leq j \leq 2n)$ be an element of Q_{4n}.

Then, $a^n.a^j = a^{n+j} = a^j.a^n \implies a^n \sim a^j, 1 \leq j \leq 2n$.

Again consider an element $\alpha = a^ib \in Q_{4n}, 1 \leq i \leq 2n$. Then,

$$\begin{aligned}
a^n\alpha &= a^{n+i}b = ba^{-(n+i)}(\because b^{-1}ab = a^{-1} \implies ab = ba^{-1}) \\
&= ba^{-i}a^{-n} \\
&= ba^{-i}a^{n}(\because a^{2n} = e \implies a^n = a^{-n}) \\
&= (a^ib)a^n = \alpha a^n
\end{aligned}$$

This implies $a^n = \alpha^{-1}a^n\alpha$, $\forall \alpha \in Q_{4n} \implies a^n \in Z(Q_{4n})$.

Therefore, $[a^n] = \{a^n\}$

Consider the elements of Q_{4n} of the form $a^i(1 \leq i \leq n)$ and $(n < i < 2n)$

We claim that $N(a^i) = \{a, a^2, a^3, \ldots, a^{2n} = e\}$

Because if $a^jb \in N(a^i)$, $1 \leq j \leq 2n$, then we must have

$$\begin{aligned} & (a^i)(a^jb) = (a^jb)(a^i) \\ \Longrightarrow \quad & a^{i+j}b = ba^{-j}a^i \\ \Longrightarrow \quad & a^{i+j} = a^{-j+i} \\ \Longrightarrow \quad & a^i = a^{-i} \\ \Longrightarrow \quad & a^{2i} = 1 = a^{2n} \\ \Longrightarrow \quad & i = n \end{aligned}$$

which is a contradiction to the fact that $1 \leq i < n$ and $n < i < 2n$.
Therefore, $N(a^i) = \{a, a^2, a^3, \ldots, a^{2n-1}, a^{2n} = e\}$
Then, $|[a^i]| = \frac{|Q_{4n}|}{|N(a^i)|} = \frac{4n}{2n} = 2$
Since $b^{-1}ab = a^{-1} = a^{2n-1}$, so $a \sim a^{2n-1}$
Again, $b^{-1}a^2b = a^{-2} = a^{2n-2}$. Therefore, $a^2 \sim a^{2n-2}$
Similarly, $a^3 \sim a^{2n-3}$, $a^4 \sim a^{2n-4}$, $\ldots$, $a^{n-1} \sim a^{n+1}$
Thus, conjugacy class containing a^i, $1 \leq i < n$, $n < i < 2n$, contains only two elements. Hence the number of conjugacy classes with only two elements is $\frac{2n-2}{2} = n-1$.

Consider the elements of Q_{4n} of the form a^ib, $1 \leq i \leq 2n$
Suppose $a^j \in N(a^ib)$, $1 \leq j \leq 2n$. Then,

$$\begin{aligned} & (a^j)(a^ib) = (a^ib)(a^j) \\ \Longrightarrow \quad & a^{j+i}b = ba^{-i}a^j = ba^{-i+j} \\ \Longrightarrow \quad & a^{i+j}b = a^{i-j}b \\ \Longrightarrow \quad & a^j = a^{-j} \\ \Longrightarrow \quad & j = n \quad or \quad j = 2n \end{aligned}$$

Hence, $\{a^n, ^{2n}\} \subseteq N(a^ib)$
Suppose $a^jb \in N(a^ib)$, $1 \leq j \leq 2n$, $j \neq i$. Then,

$$\begin{aligned} & (a^ib)(a^jb) = (a^jb)(a^ib) \\ \Longrightarrow \quad & a^ia^{-j}bb = a^ja^{-i}bb \\ \Longrightarrow \quad & a^{i-j} = a^{j-i} \\ \Longrightarrow \quad & a^{2(j-i)} = 1 = a^{2n} \\ \Longrightarrow \quad & j = i + n \end{aligned}$$

Thus, $\{a^ib, a^jb\} \subseteq N(a^ib)$, where $j = i + n$
Finally we get, $N(a^ib) = \{a^{2n} = e, a^n, a^ib, a^{i+n}b\}$

Therefore, $|[a^ib]| = \frac{|Q_{4n}|}{|N(a^ib)|} = \frac{4n}{4} = n$
Thus the conjugacy class of a^ib contains n elements.
Let us consider two elements of Q_{4n} of the form a^ib and a^jb, where $1 < i, j \leq 2n$, $i < j$.
Case 1. Suppose i and j are both even and $j - i = 2$.
Then there exists $a^{i+1}b \in Q_{4n}$ such that

$$\begin{aligned}(a^{i+1}b)^{-1}(a^ib)(a^{i+1}b) &= b^{-1}a^{-i-1}(a^ib)(a^{i+1}b)\\ &= b^{-1}(a^{-1}b)(a^{i+1}b)\\ &= b^{-1}(ba)(a^{i+1}b) \quad (\because ab = ba^{-1})\\ &= aa^{i+1}b = a^{i+2}b = a^jb\end{aligned}$$

Thus we have, $a^ib \sim a^jb$ i. e. $a^2b \sim a^4b \sim a^6b \sim \ldots \sim a^{2n-2}b \sim b$
Case 2. Suppose i and j both are odd and $j - i = 2$.
Consider two elements of the form a^ib and a^jb, where $1 \leq i, j \leq 2n$, $i < j$.
By similar argument we can show that $a^ib \sim a^jb$.
Thus we have, $ab \sim a^3b \sim a^5b \sim \ldots \sim a^{2n-3}b \sim a^{2n-1}b$
Therefore,

$$N(a^ib) = \{a^{2n} = e, a^n, a^ib, a^{i+n}b\}$$
$$N(ab) = \{e, a^n, ab, a^{n+1}b\}$$
$$N(ab) = \{e, a^n, a^2b, a^{n+2}b\}$$

Hence we have, $|[ab]| = \frac{|Q_{4n}|}{|N(ab)|} = \frac{4n}{4} = n$
and $|[a^2b]| = \frac{|Q_{4n}|}{|N(a^2b)|} = \frac{4n}{4} = n$
Therefore, the conjugacy class of ab contains n elements and they are $ab \sim a^3b \sim a^5b \sim \ldots \sim a^{2n-3}b \sim a^{2n-1}b$ and the conjugacy class of a^2b contains n elements and they are $a^2b \sim a^4b \sim a^6b \sim \ldots \sim a^{2n-2}b \sim b$. Hence a^ib and a^jb belong to different conjugacy classes when i is odd and j is even. So the total number of conjugacy classes will be $\frac{2n-2}{2} + 2 + 2 = n + 3$.
The following results are thus obtained.

Theorem 1. *The conjugacy class graph of $Q_{4n} \times Q_{4m}$ are either complete graphs or consists of complete maximal subgraphs $K_{mn+3m+3n-7}$ or $K_{mn+3m+3n+1}$, for different values of m, n and $m, n > 2$. For $m, n = 2$ the conjugacy class graph is a disconnected graph.*
Proof. The orders of conjugacy classes of Q_{4n} and Q_{4m} are respectively as follows:

$$1, 1, \underbrace{2, 2, \ldots, 2}_{n-1}, n-1, n-1$$

and

$$1, 1, \underbrace{2, 2, \ldots, 2}_{m-1}, m-1, m-1$$

There are 2 conjugacy classes of order 1, $n-1$ conjugacy classes of order 2 and 2 conjugacy classes of order $n-1$ in Q_{4n}.
Therefore, the orders of the conjugacy classes in $Q_{4n} \times Q_{4m}$ will be as follows:
$1, 1, \underbrace{2, \ldots, 2}_{2m+2n-4}, \underbrace{4, \ldots, 4}_{(m-1)(n-1)}, \underbrace{m-1, \ldots, m-1}_{4}, \underbrace{n-1, \ldots, n-1}_{4},$
$\underbrace{(n-1)(m-1), \ldots, (n-1)(m-1)}_{4}, \underbrace{2(m-1), \ldots, 2(m-1)}_{2n-2}, \underbrace{2(n-1), \ldots, 2(n-1)}_{2m-2}$

Total number of conjugacy classes in $Q_{4n} \times Q_{4m}$ is $2 + (2m+2n-4) + (m-1)(n-1) + 4 + 4 + 4 + 2n - 2 + 2m - 2 = mn + 3m + 3n + 7$. Since vertices of conjugacy class graphs are the non-central conjugacy classes, the number of vertices in $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ will be $mn + 3m + 3n + 5$.

Case 1. Both m, n are odd.
In this case, orders of the vertices will be $2, 4, m-1, n-1, (m-1)(n-1), 2(m-1), 2(n-1)$ which are all even. Hence we get a complete graph $K_{mn+3m+3n+5}$.

Case 2. Both m, n are even $(m, n > 2)$.
The orders of the vertices will be $2, 4, m-1(odd), n-1(odd), (m-1)(n-1)(odd), 2(m-1)(even), 2(n-1)(even)$.

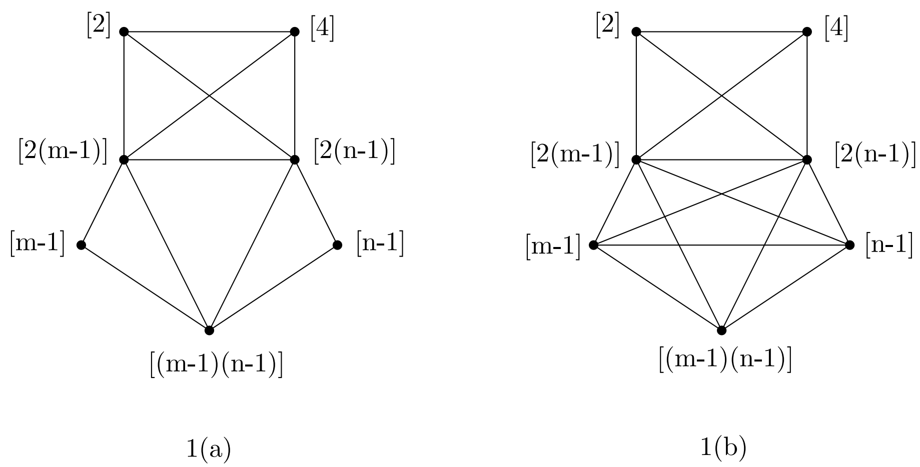
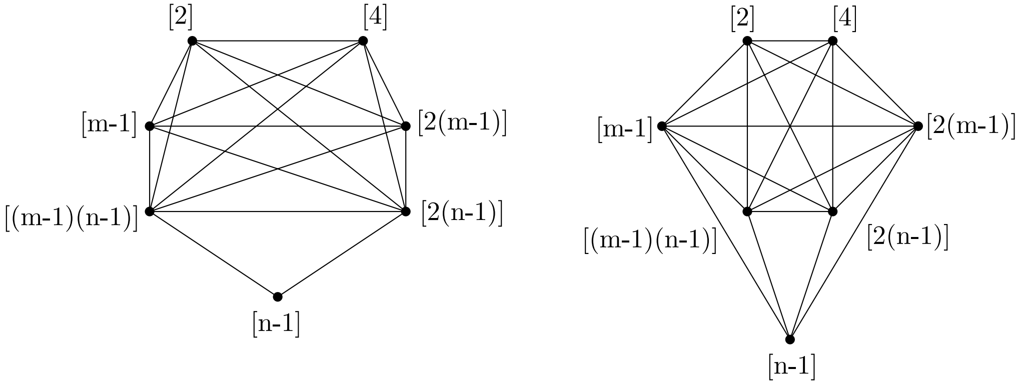


Figure 1: Conjugacy class graph of $Q_{4n} \times Q_{4m}$ (when $m, n = 0\ (mod\ 2)$)

Clearly, for $m, n = 2$ the vertices with orders $m-1$, $n-1$ and $(m-1)(n-1)$ are all isolated vertices and so the resulting conjugacy class graph is disconnected.

For $m, n > 2$, we get two different graph structures (Fig. 1) depending on whether $gcd(m-1, n-1) = 1$ or $gcd(m-1, n-1) > 1$.
If $gcd(m-1, n-1) = 1$, then $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ comprises of one complete subgraph $K_{mn+3m+3n-7}$ (for $m, n > 2$).
If $gcd(m-1, n-1) > 1$, then we get two complete subgraphs $K_{2m+2n+8}$ and $K_{mn+3m+3n-7}$, the latter being maximal in $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ $(m, n > 2)$.
Case 3. One of m, n is even and the other one odd.
Let n be even and m is odd $(n > 2)$.
Then orders of the conjugacy classes are $2, 4, m-1(even), n-1(odd), (m-1)(n-1)(even), 2(m-1)(even), 2(n-1)(even)$. The graph structures look like (Fig. 2):



2(a) 2(b)

Figure 2: Conjugacy class graph of $Q_{4n} \times Q_{4m}$ (when $m = 1\ (mod\ 2)$, $n = 0\ (mod\ 2)$)

If $gcd(m-1, n-1) = 1$, then we get two complete subgraphs K_{2m+6} and $K_{mn+3m+3n+1}$, which is also the maximal subgraph contained in $\Gamma^{cl}(Q_{4n} \times Q_{4m})$.
If $gcd(m-1, n-1) > 1$, then the two complete subgraphs are $K_{2m+2n+8}$ and $K_{mn+3m+3n+1}$. For both the cases, the complete maximal subgraph is $K_{mn+3m+3n+1}$.

Observations

Corollary 1.1. *$\Gamma^{cl}(Q_{4n} \times Q_{4m})$ is connected for all $m, n > 2$.*
Proof. The result is obvious from the graph structures shown above. If either or both $m, n = 2$, then the graph consists of isolated vertices.

Corollary 1.2. *$\Gamma^{cl}(Q_{4n} \times Q_{4m})$ is non planar for all m, n.*
Proof. For each of the three cases discussed in the previous theorem, $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ consists of subgraph K_5 for all m, n. Hence, $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ is non- planar.

Corollary 1.3. *Diameter of* $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ *is* 2.

Proof. For each case as discussed above, there are at least two vertices which are adjacent to every vertex of the graph. So, each vertex is connected to the other vertices by a path of length 2.

Corollary 1.4. *The chromatic number and clique number of* $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ *are equal and is given by:*

$$\chi(\Gamma^{cl}(Q_{4n} \times Q_{4m})) = \omega(\Gamma^{cl}(Q_{4n} \times Q_{4m}))$$
$$= \begin{cases} mn + 3m + 3n + 5, \text{ if } m\,, n \ \equiv 1 \ (mod\ 2) \\ mn + 3m + 3n - 7, \text{ if } m\,, n \ \equiv 0 \ (mod\ 2) \\ mn + 3m + 3n + 1, \text{ if } n \ \equiv 0 \ (mod\ 2), \ m \ \equiv 1 \ (mod\ 2) \end{cases}$$

Proof. The result is obvious from **Theorem 1**.

Corollary 1.5. *The independence number of* $\Gamma^{cl}(Q_{4n} \times Q_{4m})$ *is given by:*

$$\alpha(\Gamma^{cl}(Q_{4n} \times Q_{4m})) = \begin{cases} 1, & \text{if } m\,, n \ \equiv 1 \ (mod\ 2) \\ 3 \ \ or \ \ 2, & \text{if } m\,, n \ \equiv 0 \ (mod\ 2) \\ 2, & \text{if } n \ \equiv 0 \ (mod\ 2), \ m \ \equiv 1 \ (mod\ 2) \end{cases}$$

For even values of m, n, the independence number is 3 if $gcd(m-1, n-1) = 1$ and 2 if $gcd(m-1, n-1) > 1$.

Theorem 2. *The conjugacy class graph of* $S_n \times S_m$ *consists of complete maximal subgraphs* K_{rs-4} *for* $3 \le n, m \le 4$ *and* K_{rs-3} *for* $3 \le n \le 4$, $m = 2$, *where* r *ans* s *are the number of conjugacy classes of* S_n *and* S_m *respectively.*

Proof. The number of conjugacy classes of S_n is the number of partitions of n, denoted by $p(n)$. The size of the conjugacy classes corresponding to each partition are given by:

$$|Cl| = \frac{n!}{\prod (i)^{a_i}(a_i!)}$$

where a_i is the number of times i occurs in the partition of n. By manual calculations, we now find the conjugacy classes and hence the conjugacy class graphs of $S_n \times S_m$ $(3 \le m, n \le 5)$.

(i) $S_3 \times S_3$

The conjugacy classes of S_3 are of orders $1, 2, 3$. Therefore, the conjugacy classes of $S_3 \times S_3$ will be of orders $1, 2, 3, 4, 6, 9$. Since the vertices of conjugacy class graphs are the non central conjugacy classes, the number of vertices in $\Gamma^{cl}(S_3 \times S_3)$ is $9 - 1 = 8$. There is one vertex of order 1, two vertices of order 2, two vertices of order 3, one vertex of order 4, two vertices of order 6 and one vertex of order 9 in $\Gamma^{cl}(S_3 \times S_3)$. The vertices with orders $2, 4, 6$ forms a complete subgraph K_5 and the vertices with orders $3, 6, 9$ forms another complete subgraph K_5 in $\Gamma^{cl}(S_3 \times S_3)$, both of the subgraphs having two vertices with order 6 as the connecting vertices. The conjugacy class graph of $S_3 \times S_3$ is shown below (Fig. 3):

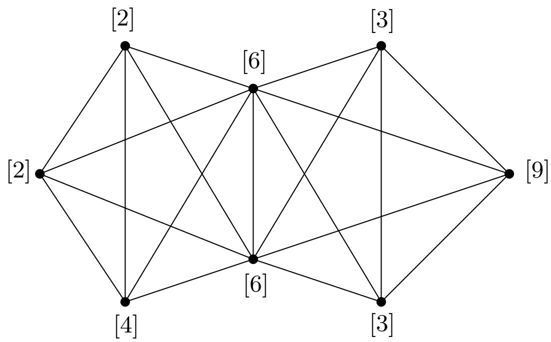


Figure 3: Conjugacy class graph of $S_3 \times S_3$

(ii) $S_3 \times S_4$
The orders of the conjugacy classes of S_3 are $1, 2, 3$ and that of S_4 are $1, 3, 6, 6, 8$. The orders of the 15 conjugacy classes of $S_3 \times S_4$ are $1, 2, 3, 3, 6, 6, 6, 8, 9, 12, 12, 16, 18, 18, 24$. So, the number of vertices in $\Gamma^{cl}(S_3 \times S_4)$ is 14. The vertices with even orders form a complete subgraph K_{11} while the vertices having orders as multiples of 3 form another complete subgraph K_{11}.
(iii) $S_4 \times S_4$
In this case, orders of the conjugacy classes are $1, 3, 6, 8, 9, 18, 24, 36, 48, 64$ and the number of conjugacy classes is 25. The vertices with even orders form a complete subgraph K_{21} and the vertices having orders as multiples of 3 form another subgraph K_{21}.
(iv) $S_3 \times S_5$
The 7 conjugacy classes of S_5 consists of orders $1, 10, 15, 20, 20, 24, 30$. So the number of vertices in $\Gamma^{cl}(S_3 \times S_5)$ is 20 and the orders of the conjugacy classes are $1, 2, 3, 10, 15, 20, 20, 20, 24, 30, 30, 30, 40, 40, 45, 48, 60, 60, 60, 72, 90$. The vertices with orders $10, 15, 20, 24, 30, 40, 45, 48, 60, 72, 90$ forms the maximal complete subgraph K_{18} in $\Gamma^{cl}(S_3 \times S_5)$.
(v) $S_4 \times S_5$
The 5 conjugacy classes of S_4 and the 7 conjugacy classes of S_5 have orders $1, 3, 6, 6, 8$ and $1, 10, 15, 20, 20, 24, 30$ respectively. Hence the number of vertices in $\Gamma^{cl}(S_4 \times S_5)$ is 34. The vertices having orders as multiples of 3 (except the

vertex with order 3) or 5 or both forms a complete subgraph K_{32}.

Conjecture. *$\Gamma^{cl}(S_n \times S_m)$ is a complete graph for $n, m \geq 5$.*
As a consequence of the previous theorem, the following results thus follow.

Corollary 3.1. *$\Gamma^{cl}(S_n \times S_m)$ is connected for all $m, n \geq 3$.*

Corollary 3.2. *$\Gamma^{cl}(S_n \times S_m)$ is non planar for all $m, n \geq 3$.*

Corollary 3.3. *Given r ans s are the number of conjugacy classes of S_n and S_m respectively, then the clique number of $\Gamma^{cl}(S_n \times S_m)$ is given as follows:*

$$\omega(\Gamma^{cl}(S_n \times S_m)) = \begin{cases} rs - 4, & 3 \leq m, n \leq 4 \\ rs - 3, & 3 \leq n \leq 4, m = 5 \\ rs - 1, & m, n \geq 5 \end{cases}$$

4. Commutativity Degree and Generalized Conjugacy Class Graph

In this section, we derive the generalized conjugacy class graphs for the dihedral group and the generalized quaternion group, along with analyzing the commutativity degree in relation to the conjugacy classes of these groups.

Theorem 3. *The generalized conjugacy class graph of D_n (for odd values of n) is the complete graph K_2.*

Proof. For odd values of n, the dihedral group D_n consists of $2n$ elements and $\frac{n+3}{2}$ conjugacy classes which are $\{e\}$, $\{a, a^{n-1}\}$, $\{a^2, a^{n-2}\}$, $\ldots$, $\{a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}\}$, $\{b, ab, a^2b, a^3b, \ldots, a^{n-1}b\}$.
By definition, $\Omega = \{(x, y) \in D_n \times D_n | xy = yx, x \neq y, lcm(|x|, |y|) = 2\}$ and the orders of the elements of D_n are such that there is one element of order 1, n elements of order 2 and $\phi(d)$ elements of order d, where $d|n$. From the definition of Ω, we can infer that we only need elements with order 1 and 2 for the construction of Ω. Clearly, the element e has order 1 and the elements of the conjugacy class $G = \{b, ab, a^2b, a^3b, \ldots, a^{n-1}b\}$ have order 2. The elements of the rest conjugacy classes have order greater than 2. Now, for any $a^ib, a^jb \in G$, $(a^ib)(a^jb) \neq (a^jb)(a^ib)$. Thus we can now compute the elements of Ω as follows:
$\Omega = \{(e, b), (e, ab), (e, a^2b), \ldots, (e, a^{n-1}b), (b, e), (ab, e), (a^2b, e), \ldots, (a^{n-1}b, e)\}$
The orbits are calculated as:

$$\begin{aligned} \omega_1 &= O(e, b) = \{(e, b), (e, ab), (e, a^2b), \ldots, (e, a^{n-1}b)\} \\ &= O(e, ab) \\ &= O(e, a^2b) \\ &\ \vdots \\ &= O(e, a^{n-1}b) \end{aligned}$$

$$\begin{aligned}\omega_2 = O(b,e) &= \{(b,e),(ab,e),(a^2b,e),\dots,(a^{n-1}b,e)\}\\ &= O(ab,e)\\ &= O(a^2b,e)\\ &\ \vdots\\ &= O(a^{n-1}b,e)\end{aligned}$$

These are the only two non-central orbits of Ω for D_n (odd n) and by definition $K(\Omega) = 2$.
Also, ω_1 and ω_2 have n elements each, so $gcd(|\omega_1|, |\omega_2|) > 1$. Hence the two vertices in $\Gamma^{\Omega_c}_{D_n}$ are adjacent resulting in the complete graph K_2.

Theorem 4. *The generalized conjugacy class graph of D_n (for even values of n) is the complete graph K_{10}.*
Proof. For even values of n, the dihedral group D_n of order $2n$ has $\frac{n+6}{2}$ conjugacy classes which are as shown below:
$\{e\}$, $\{a^{\frac{n}{2}}\}$, $\{a, a^{n-1}\}$, $\{a^2, a^{n-2}\}$, ... , $\{a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}\}$, $\{b, a^2b, a^4b, \dots, a^{n-2}b\}$, $\{ab, a^3b, a^5b, \dots, a^{n-1}b\}$
The orders of the elements of D_n are such that there is one element of order 1, $n + 1$ elements of order 2 and $\phi(d)$ elements of order d, where $d|n$. Since we need all x, y such that $lcm(|x|, |y|) = 2$ $(x, y \in D_n)$, we are only concerned with those elements whose orders are 1 and 2. Such elements are e, $a^{\frac{n}{2}}$ and the elements of the conjugacy classes $G_1 = \{b, a^2b, \dots, a^{n-2}b\}$ and $G_2 = \{ab, a^3b, \dots, a^{n-1}b\}$. None of the elements of G_1 commute with each other, so is the case for G_2.
Now, the normalizer of $a^ib \in D_n$ is given by,

$$N(a^ib) = \{e, a^{\frac{n}{2}}, a^ib, a^{i+\frac{n}{2}}\},\ 1 \le i \le n$$

We can now compute Ω as follows:

$$\begin{aligned}\Omega = \{&(e,a^{\frac{n}{2}}),(e,b),(e,ab),\dots,(e,a^{n-2}b),(e,a^{n-1}b),\\ &(a^{\frac{n}{2}},e),(b,e),(ab,e),\dots,(a^{n-2}b,e),(a^{n-1}b,e),\\ &(a^{\frac{n}{2}},b),(a^{\frac{n}{2}},ab),(a^{\frac{n}{2}},a^2b),\dots,(a^{\frac{n}{2}},a^{n-2}b),(a^{\frac{n}{2}},a^{n-1}b),\\ &(b,a^{\frac{n}{2}}),(ab,a^{\frac{n}{2}}),(a^2b,a^{\frac{n}{2}}),\dots,(,a^{n-2}b,a^{\frac{n}{2}}),(a^{n-1}b,a^{\frac{n}{2}}),\\ &(ab,a^{1+\frac{n}{2}}b),(a^{1+\frac{n}{2}}b,ab),\dots,(a^{\frac{n}{2}-1}b,a^{n-1}b),(a^{n-1}b,a^{\frac{n}{2}-1}b)\}\end{aligned}$$

The orbits are calculated as follows:
$\omega_1 = O(e, a^{\frac{n}{2}}) = \{(e, a^{\frac{n}{2}})\}$

$\omega_2 = O(a^{\frac{n}{2}}, e) = \{(a^{\frac{n}{2}}, e)\}$

$\omega_3 = O(e, b) = \{(e, b), (e, a^2b), (e, a^4b), \ldots, (e, a^{n-2}b)\}$
$= O(e, a^2b) = O(e, a^4b) = \ldots = O(e, a^{n-2}b)$

$\omega_4 = O(b, e) = \{(b, e), (a^2b, e), (a^4b, e), \ldots, (a^{n-2}b, e)\}$
$= O(a^2b, e) = O(a^4b, e) = \ldots = O(a^{n-2}b, e)$

$\omega_5 = O(e, ab) = \{(e, ab), (e, a^3b), (e, a^5b), \ldots, (e, a^{n-1}b)\}$
$= O(e, a^3b) = O(e, a^5b) = \ldots = O(e, a^{n-1}b)$

$\omega_6 = O(ab, e) = \{(ab, e), (a^3b, e), (a^5b, e), \ldots, (a^{n-1}b, e)\}$
$= O(a^3b, e) = O(a^5b, e) = \ldots = O(a^{n-1}b, e)$

$\omega_7 = O(b, a^{\frac{n}{2}}) = \{(b, a^{\frac{n}{2}}), (a^2b, a^{\frac{n}{2}}), (a^4b, a^{\frac{n}{2}}), \ldots, (a^{n-2}b, a^{\frac{n}{2}})\}$
$= O(a^2b, a^{\frac{n}{2}}) = O(a^4b, a^{\frac{n}{2}}) = \ldots = O(a^{n-2}b, a^{\frac{n}{2}})$

$\omega_8 = O(a^{\frac{n}{2}}, b) = \{(a^{\frac{n}{2}}, b), (a^{\frac{n}{2}}, a^2b), (a^{\frac{n}{2}}, a^4b), \ldots, (a^{\frac{n}{2}}, a^{n-2}b)\}$
$= O(a^{\frac{n}{2}}, a^2b) = O(a^{\frac{n}{2}}, a^4b) = \ldots = O(a^{\frac{n}{2}}, a^{n-2}b)$

$\omega_9 = O(ab, a^{\frac{n}{2}}) = \{(ab, a^{\frac{n}{2}}), (a^3b, a^{\frac{n}{2}}), (a^5b, a^{\frac{n}{2}}), \ldots, (a^{n-1}b, a^{\frac{n}{2}})\}$
$= O(a^3b, a^{\frac{n}{2}}) = O(a^5b, a^{\frac{n}{2}}) = \ldots = O(a^{n-1}b, a^{\frac{n}{2}})$

$\omega_{10} = O(a^{\frac{n}{2}}, ab) = \{(a^{\frac{n}{2}}, ab), (a^{\frac{n}{2}}, a^3b), (a^{\frac{n}{2}}, a^5b), \ldots, (a^{\frac{n}{2}}, a^{n-1}b)\}$
$= O(a^{\frac{n}{2}}, a^3b) = O(a^{\frac{n}{2}}, a^5b) = \ldots = O(a^{\frac{n}{2}}, a^{n-1}b)$

$\omega_{11} = O(b, a^{\frac{n}{2}}b) = \{(b, a^{\frac{n}{2}}b), (a^2b, a^{\frac{n}{2}+2}b), \ldots, (a^{\frac{n}{2}-1}b, a^{n-1}b)\}$
$= O(a^2b, a^{\frac{n}{2}+2}b) = \ldots = O(a^{\frac{n}{2}-1}b, a^{n-1}b)$

$\omega_{12} = O(ab, a^{\frac{n}{2}+1}b) = \{(ab, a^{\frac{n}{2}+1}b), (a^3b, a^{\frac{n}{2}+3}b), \ldots, (a^{\frac{n}{2}-2}b, a^{n-2}b)\}$
$= O(a^3b, a^{\frac{n}{2}+3}b) = \ldots = O(a^{\frac{n}{2}-2}b, a^{n-2}b)$

These are the only 12 orbits of Ω and hence $K(\Omega) = 12$. Since the generalized conjugacy class graph do not contain the central orbits, so the number of vertices in $\Gamma_{D_n}^{\Omega_c}$ is $12 - 2 = 10$. Moreover, $gcd(|\Omega_j|, |\Omega_j|) > 1$ except for Ω_1 and Ω_2, so the resulting graph is the complete graph K_{10}.

Theorem 5. *Let D_n be the dihedral group of order $2n$ and D_n acts on Ω by conjugation. Then the probability that an element of D_n fixes the set Ω is given by:*

$$P_{D_n}(\Omega) = \begin{cases} \frac{1}{n}, & n = 1(mod2) \\ \frac{12}{5n+2}, & n = 0(mod2) \end{cases}$$

Proof. When n takes odd values, the set Ω is given by:
$\Omega = \{(e, b), (e, ab), (e, a^2b), \ldots, (e, a^{n-1}b), (b, e), (ab, e), (a^2b, e), \ldots, (a^{n-1}b, e)\}$

There are $2n$ elements in Ω and exactly two orbits of Ω. By definition,

$$P_{D_n}(\Omega) = \frac{2}{2n} = \frac{1}{n}$$

Again for even values of n, the set Ω is given by:

$$\begin{aligned}\Omega = \{&(e, a^{\frac{n}{2}}), (e, b), (e, ab), \ldots, (e, a^{n-2}b), (e, a^{n-1}b),\\ &(a^{\frac{n}{2}}, e), (b, e), (ab, e), \ldots, (a^{n-2}b, e), (a^{n-1}b, e),\\ &(a^{\frac{n}{2}}, b), (a^{\frac{n}{2}}, ab), (a^{\frac{n}{2}}, a^2b), \ldots, (a^{\frac{n}{2}}, a^{n-2}b), (a^{\frac{n}{2}}, a^{n-1}b),\\ &(b, a^{\frac{n}{2}}), (ab, a^{\frac{n}{2}}), (a^2b, a^{\frac{n}{2}}), \ldots, (, a^{n-2}b, a^{\frac{n}{2}}), (a^{n-1}b, a^{\frac{n}{2}}),\\ &(ab, a^{1+\frac{n}{2}}b), (a^{1+\frac{n}{2}}b, ab), \ldots, (a^{\frac{n}{2}-1}b, a^{n-1}b), (a^{n-1}b, a^{\frac{n}{2}-1}b)\}\end{aligned}$$

There are exactly $5n + 2$ elements in Ω and 12 orbits of Ω. Hence by definition of $P(G)$ we have,

$$P_{D_n}(\Omega) = \frac{12}{5n + 2}$$

Theorem 6. *The generalized conjugacy class graph of Q_{4n} is the null graph.*

Proof. The conjugacy classes of Q_{4n} are $\{e\}$, $\{a^n\}$, $\{a, a^{2n-1}\}$, $\{a^2, a^{2n-2}\}$, $\ldots$, $\{a^{n-1}, a^{n+1}\}$, $\{b, a^2b, a^4b, \ldots, a^{2n-2}b\}$, $\{ab, a^3b, a^5b, \ldots, a^{2n-1}b\}$. We know that elements of the form $a^ib \in Q_{4n}$ $(1 \le i \le 2n)$ has order 4 and the order of elements of the form a^i is a multiple of n (follows from standard cyclic group theory), except for the elements e and a^n which have orders 1 and 2 respectively. In that case $lcm(|a^n|, |a^i|) \ge 3$, $i \ne n$, $1 < i \le 2n$.

Thus, $\Omega = \{(e, a^n), (a^n, e)\}$. The orbits are calculated as:

$\omega_1 = O(e, a^n) = \{(e, a^n)\}$ and $\omega_2 = O(a^n, e) = \{(a^n, e)\}$

These are the only two orbits of Ω and hence $K(\Omega) = 2$. The vertices of generalized conjugacy class graph are the non central orbits of Ω and both the orbits being central orbits, so $\Gamma^{\Omega_c}_{Q_{4n}}$ is a null graph.

Theorem 7. *Let Q_{4n} be the generalized quaternion group of order $4n$ and Q_{4n} acts on Ω by conjugation. Then the probability that an element of Q_{4n} fixes the set Ω is 1.*

Proof. For Q_{4n}, the set Ω is given by:

$$\Omega = \{(e, a^n), (a^n, e)\}$$

and the only two orbits are $\omega_1 = O(e, a^n) = \{(e, a^n)\}$ and $\omega_2 = O(a^n, e) = \{(a^n, e)\}$. There are exactly 2 elements in Ω and 2 orbits of Ω. Hence by definition of $P(G)$ we have,

$$P_{Q_{4n}}(\Omega) = \frac{2}{2} = 1$$

Example 1. Consider D_3. Elements of D_3 are e, a, a^2, b, ab, a^2b and the orders of the elements are given as:
$|e| = 1$, $|a| = 3$, $|a^2| = 3$, $|b| = 2$, $|ab| = 2$, $|a^2b| = 2$.
$\Omega = \{(e, b), (e, ab), (e, a^2b), (b, e), (ab, e), (a^2b, e)\}$
$\omega_1 = O(e, b) = \{g(e, b)g^{-1} | g \in D_3\}$. Now, $e(e, b)e^{-1} = (e, b)$
$a(e, b)a^{-1} = (e, a^2b)$
$(a^2)(e, b)(a^2)^{-1} = (e, ab)$
$b(e, b)b^{-1} = (e, b)$
$(ab)(e, b)(ab)^{-1} = (e, a^2b)$, $\quad (a^2b)(e, b)(a^2b)^{-1} = (e, ab)$
Thus, $\omega_1 = \{(e, b), (e, ab), (e, a^2b)\}$. Similarly, $\omega_2 = O(b, e) = \{(b, e), (ab, e), (a^2b, e)\}$. These are the only two orbits of D_3, both of which are non-central. Thus, the resulting graph is the complete graph K_2.

Example 2. Consider $Q_{4.2}$. Elements of $Q_{4.2}$ are $e, a, a^2, a^3, b, ab, a^2b, a^3b$ and the orders are given as:
$|e| = 1$, $|a| = 4$, $|a^2| = 2$, $|a^3| = 4$, $|b| = 4$, $|ab| = 4$, $|a^2b| = 4$, $|a^3b| = 4$.
$\Omega = \{(e, a^2), (a^2, e)\}$, $\omega_1 = O(e, a^2) = \{(e, a^2)\}$, $\omega_2 = O(a^2, e) = \{(a^2, e)\}$.
Both the orbits are central orbits, therefore there are no vertices in $\Gamma^{\Omega_c}_{Q_{4n}}$.

Remark. *It is known that the commutativity degree of abelian groups is 1, meaning that in such groups, every element commutes with every other. The orbits in the generalized conjugacy class graphs of Q_{4n} are central, since the associated probability is also found to be 1. Therefore, even without examining the individual elements within each orbit, we can infer the abelian nature of the group formed by the orbits purely from a probabilistic standpoint.*

5. Conclusion

In this paper, we observe that the conjugacy class graph of $Q_{4n} \times Q_{4m}$ $(m, n > 2)$ is a complete graph when m, n are odd, and consists of complete maximal subgraphs when either one of m, n or both are even. Also, the conjugacy class graph of $S_n \times S_m$ $(m, n \geq 3)$ consists of complete maximal subgraphs when $n < 5$ and $m \geq 5$, while the conjugacy class graph of $S_n \times S_m$ is conjectured to be a complete graph for $n, m \geq 5$. The generalized conjugacy class graph of D_n is isomorphic to the complete graph K_2 for odd n and isomorphic to the complete graph K_{10} for even n. The generalized conjugacy class graph for Q_{4n} is a null graph. The probability that an element of D_n fixes the set Ω is obtained and it is seen that the probability decreases as the value of n increases, while the probability obtained in the case of generalized quaternion groups remains constant.

Future Scope

Efforts are currently underway to compute the conjugacy classes associated with the direct product of dihedral and symmetric groups, as well as for the generalized quaternion group and the symmetric group. The complements of graphs and the line graphs associated with direct products across various groups could be explored. In light of the conjectured completeness of the conjugacy class graphs for the group $S_n \times S_m$, one may attempt to prove this conjecture. If we examine the conjugacy class structure of the alternating group as a subgroup of S_n, we observe that the conjugacy class graphs of these groups, as well as of their direct products, are complete graphs. Since, A_n is simple for $n \geq 5$, it would be intriguing to investigate whether all finite non-abelian simple groups possess complete conjugacy class graphs.

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